

$QL(\mathbb{C}^n)$  DETERMINES  $n$ 

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**Abstract.** This addendum to [2] shows that the set of tautological quantum logical propositional formulas for a finite dimensional vector space  $\mathbb{C}^n$  is different for every  $n$ , affirmatively answering a question posed therein.

The paper [2] explored the properties of Birkhoff and Von Neumann's propositional quantum logic (see [1]) as modelled by finite dimensional Hilbert spaces. One question asked in [2] is whether the set of tautological propositional formulas uniquely determines the dimension of the underlying vector space. A partial answer was given, namely that  $\mathbb{C}^n$  and  $\mathbb{C}^{2n}$  give different sets of tautologies. This note gives a full answer to the question.

For our purposes, propositional formulas consist of alphabet symbols, perentheses, and the symbols meet ( $\wedge$ ), join ( $\vee$ ), orthocomplement ( $\neg$ ), top ( $\top$ ), and bottom ( $\perp$ ). The well formed formulas are the same as those of propositional boolean logic. The symbol  $\top$  is interpreted as a finite dimensional Hilbert space,  $\perp$  is the trivial subspace, alphabet symbols are variables standing for vector subspaces of  $\top$ ,  $\wedge$  is intersection,  $\vee$  is span of union, and  $\neg$  is orthogonal complement in  $\top$ . With these operations, the set of subspaces of  $\top$  forms a bounded modular ortholattice.

Let  $\bar{v} = v_1, \dots, v_k$  be a list of alphabet symbols and let  $\bar{S} = S_1, \dots, S_k$  be a collection of subspaces of a finite dimensional Hilbert space  $U$ . Given a well formed formula  $\phi(\bar{v})$ , the valuation  $\Xi_U(\phi(\bar{v}), \bar{S})$  is the subspace resulting from instantiating each  $v_i$  with the subspace  $S_i$  and performing the operations described by  $\phi$  with universal space  $U$ . As a shorthand the valuation may be implicit; for example if  $S$  and  $T$  are subspaces of  $U$  then  $\Xi_U(v \wedge w, S, T)$  is abbreviated  $S \wedge T$ , and  $U$  is inferred from context.

DEFINITION 1. Let  $\phi(\bar{v})$  be a well formed formula. Define  $\bar{d}_\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$\bar{d}_{\phi(\bar{v})}(n) = \max_{\bar{S}}(\dim(\Xi_{\mathbb{C}^n}(\phi(\bar{v}), \bar{S}))).$$

DEFINITION 2. A well formed formula  $\phi(\bar{v})$  is a *tautology* in  $\mathbb{C}^n$  if  $\bar{d}_\phi(n) = 0$ .

DEFINITION 3.  $QL(\mathbb{C}^n)$  is the set of tautologies when  $\top = \mathbb{C}^n$ .

The goal is to establish the following:

THEOREM 1.  $m < n \Rightarrow QL(\mathbb{C}^n) \subsetneq QL(\mathbb{C}^m)$ .

In [2] it was shown that  $QL(\mathbb{C}^{n+1}) \subset QL(\mathbb{C}^n)$ , and  $QL(\mathbb{C}^n) \neq QL(\mathbb{C}^{2n})$ . A formula  $\phi_1$  was constructed such that  $\bar{d}_{\phi_1} = \lfloor \frac{n}{2} \rfloor$ . From  $\phi_1$  formulas  $\phi_k$  were constructed in stages such that  $\bar{d}_{\phi_k} = \lfloor \frac{\bar{d}_{\phi_{k-1}}}{2} \rfloor$ . Thus  $\phi_k$  is a tautology in  $QL(\mathbb{C}^n)$  iff  $\log_2(n) < k$ . Each  $\phi_k = \phi_1|_{\phi_{k-1}}$ , which is defined as follows:<sup>1</sup>

DEFINITION 4. Suppose that  $\alpha(\bar{u})$  is a formula in  $k$  variables  $\bar{u} = u_1, \dots, u_k$ . For convenience, use De Morgan's laws to replace  $\alpha$  with an equivalent formula, also called  $\alpha$ , in which all negations are negations of atomic variable symbols. Given a second formula  $\beta(\bar{v})$ , let  $\alpha(\bar{u})|_{\beta(\bar{v})}$  denote the modification of  $\alpha$  such that each unnegated instance of  $u_i$  is replaced with  $u_i \wedge \beta(\bar{v})$  and each instance of  $\neg u_i$  is replaced with  $\neg(u_i \wedge \beta(\bar{v})) \wedge \beta(\bar{v})$ .

LEMMA 2. Let  $\bar{u} = u_1, \dots, u_{k_u}$  and  $\bar{v} = v_1, \dots, v_{k_v}$  be lists of variables,  $\bar{S} = S_1, \dots, S_{k_u}$  and  $\bar{T} = T_1, \dots, T_{k_v}$  lists of subspaces of  $\mathbb{C}^n$ , and  $\alpha(\bar{u})$  and  $\beta(\bar{v})$  formulas. Let  $\bar{P} = P_1, \dots, P_{k_u}$  such that  $P_i = S_i \wedge \Xi_{\mathbb{C}^n}(\beta(\bar{v}), \bar{T})$ . Then the following holds:

$$\Xi_{\mathbb{C}^n}(\alpha(\bar{u})|_{\beta(\bar{v})}, \bar{S}, \bar{T}) = \Xi_{\Xi_{\mathbb{C}^n}(\beta(\bar{v}), \bar{T})}(\alpha(\bar{u}), \bar{P})$$

PROOF. The construction procedure gives the result for atomic formulas and their negations. Since unions and intersections are not changed by inclusion into a larger universal space, the result follows by structural induction.  $\dashv$

COROLLARY 3. If  $\bar{d}_{\alpha(\bar{u})} = f$ , and  $\bar{d}_{\beta(\bar{v})} = g$ , then  $\bar{d}_{\alpha(\bar{u})|_{\beta(\bar{v})}} = f \circ g$ . In particular,  $\alpha(\bar{u})|_{\beta(\bar{v})}$  is a tautology in  $\mathbb{C}^n$  iff  $\alpha(\bar{u})$  is a tautology in  $\mathbb{C}^{g(n)}$ .

Because the function  $n \rightarrow \lfloor \frac{n}{2} \rfloor$  is not injective, none of the formulas constructed in [2] distinguish dimensions between  $2^k$  and  $2^{k+1} - 1$ . To overcome this limitation, suppose  $2^k \leq m < n \leq 2^{k+1} - 1$  for some  $k$ , and assume there exists a formula  $\alpha$  such that  $\bar{d}_\alpha = \lfloor \frac{n}{2} \rfloor$ . Construct a formula  $\phi$  in stages, starting with  $\phi_0 = \top$ .

<sup>1</sup>This definition corrects a small mistake in the definition given in [2].

Suppose it is the beginning of stage  $s$ , and  $\bar{d}_{\phi_{s-1}}(m) < \bar{d}_{\phi_{s-1}}(n)$ . If  $m$  is odd, or  $\bar{d}_{\phi_{s-1}}(n) - \bar{d}_{\phi_{s-1}}(m) > 1$ , define  $\phi_s(\bar{u}, \bar{v}) = \alpha(\bar{u})|_{\phi_{s-1}(\bar{v})}$ . Then by Corollary 3,  $\bar{d}_{\phi_s} = \lfloor \frac{\bar{d}_{\phi_{s-1}}}{2} \rfloor$ , and  $\bar{d}_{\phi_s}(m) < \bar{d}_{\phi_s}(n)$ . If, on the other hand, there is some  $l$  such that  $\bar{d}_{\phi_{s-1}}(m) = 2l$  and  $\bar{d}_{\phi_{s-1}}(n) = 2l + 1$ , it will be shown that there is a formula  $\beta_l$ , which depends on  $l$ , such that  $\bar{d}_{\beta_l}(2l) = l$  and  $\bar{d}_{\beta_l}(2l + 1) = l + 1$ . Then define  $\phi_s(\bar{u}, \bar{v}) = \beta_l(\bar{u})|_{\phi_{s-1}(\bar{v})}$ . Then by Corollary 3,

$$\bar{d}_{\phi_s}(m) = \bar{d}_{\beta_l}(\bar{d}_{\phi_{s-1}}(m)) = l < l + 1 = \bar{d}_{\beta_l}(\bar{d}_{\phi_{s-1}}(n)) = \bar{d}_{\phi_s}(n).$$

Since at each stage  $\bar{d}_{\phi_s}(m) = \lfloor \frac{\bar{d}_{\phi_{s-1}}(m)}{2} \rfloor$  and  $\bar{d}_{\phi_s}(m) < \bar{d}_{\phi_s}(n)$ , the construction procedure must eventually give  $\phi_s$  such that  $\bar{d}_{\phi_s}(m) = 0$  but  $\bar{d}_{\phi_s}(n) > 0$ . It remains only to construct  $\alpha$  and  $\beta_l$ . A suitable formula for  $\alpha$  was given in [2], but here a new  $\alpha$  will be constructed and then modified to give  $\beta_l$ .

Let  $P_c$  denote the linear operator that projects onto the subspace given by the variable  $c$ . The following formula evaluates to the image of  $P_b \circ P_a$ :

$$P(a, b) = (a \vee \neg b) \wedge b.$$

In a distributive lattice  $P(a, b) = a \wedge b$ , but in a modular lattice one only has  $P(a, b) \geq a \wedge b$ . The following are true when the lattice is subspaces of  $\mathbb{C}^n$ :

LEMMA 4. *for all subspaces  $S$  and  $T$ , the following hold:*

1.  $\dim(P(S, T)) = \dim(P(T, S)) \leq \min(\dim(S), \dim(T))$ ,
2.  $S \wedge P(S, T) = T \wedge P(T, S) = P(S, T) \wedge P(T, S) = S \wedge T$ .

The proof is easy and omitted.

Define the formula

$$\alpha(a, b) = P(b, a) \wedge \neg(a \wedge b).$$

When the distributive law holds  $\alpha$  is a tautology, but  $\alpha$  is not a tautology in  $\mathbb{C}^n$  for  $n \geq 2$ .

LEMMA 5. *The formula  $\alpha(a, b)$  has  $\bar{d}_\alpha = \lfloor \frac{n}{2} \rfloor$  and for spaces  $S$  and  $T$ ,  $\dim(\Xi_{\mathbb{C}^n}(\alpha(a, b), S, T)) = \frac{n}{2}$  iff the following hold:*

1.  $n$  is even,
2.  $\dim(S) = \dim(T) = \frac{n}{2}$ ,
3.  $S \wedge T = \perp$ ,
4.  $S \wedge \neg T = \perp$ .

PROOF. It is easy to verify that the above conditions on  $S$  and  $T$  give dimension  $\frac{n}{2}$ . For the other direction, if  $\min(\dim(S), \dim(T)) < \frac{n}{2}$ ,  $\dim(\alpha(S, T)) \leq \dim(P(S, T)) < \frac{n}{2}$ . Also, since  $S \wedge T \subset P(T, S)$ , one gets the following:

$$\begin{aligned}
(1) \quad & \dim(\alpha(S, T)) = \dim(P(T, S)) - \dim(S \wedge T) \\
(2) \quad & \leq \min(\dim(T), \dim(S)) - \dim(S \wedge T) \\
(3) \quad & \leq \min(\dim(T), \dim(S)) - \dim(S) - \dim(T) + \dim(\top) \\
(4) \quad & = \dim(\top) - \max(\dim(S), \dim(T)).
\end{aligned}$$

Therefore, if  $\dim(\alpha(S, T)) = \frac{n}{2}$ ,  $\dim(S) = \dim(T) = \frac{n}{2}$ . Thus  $\dim(P(T, S)) \leq \frac{n}{2}$ , so line 1 implies that  $\dim(P(T, S)) = \frac{n}{2}$  and  $\dim(S \wedge T) = 0$ . Since  $\dim(P(T, S)) = \frac{n}{2}$ ,  $\dim(S \wedge \neg T) = 0$ .  $\dashv$

COROLLARY 6.  $\dim(\alpha(S, T)) = \frac{n}{2} \Rightarrow \alpha(S, T) = P(T, S)$ .

To define  $\beta_l$ , restrict  $\alpha$  to itself  $\lfloor \log_2(l) \rfloor - 1$  times to obtain a formula  $\gamma$  such that  $\bar{d}_\gamma(2l) = \bar{d}_\gamma(2l + 1) = 1$ . Define  $\tilde{\beta}_l(a, b, \bar{c}) = \neg(P(b, a) \vee P(a, b)) \wedge \gamma(\bar{c})$  and  $\beta_l(a, b, \bar{c}) = \tilde{\beta}_l(a, b, \bar{c}) \vee \alpha(a, b)$ .

LEMMA 7. *The formula  $\beta_l$  satisfies  $\bar{d}_{\beta_l}(2l) = l$  and  $\bar{d}_{\beta_l}(2l + 1) = l + 1$ .*

PROOF. Clearly,  $\bar{d}_{\tilde{\beta}_l}(2l) = \bar{d}_{\tilde{\beta}_l}(2l + 1) = 1$ . The conditions of Lemma 5 imply that  $\bar{d}_{\beta_l}(2l) = \bar{d}_\alpha(2l) = l$ , while  $\bar{d}_{\beta_l}(2l + 1) = \bar{d}_\alpha(2l + 1) + 1 = l + 1$ .  $\dashv$

#### REFERENCES

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